TOROIDAL BUBBLE MOTION IN UNBOUNDED INCOMPRESSIBLE FLUID

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The study of toroidal vortex ring motion is of interest primarily because at certain conditions the initially spherical gas bubble takes the form of a toroidal bubble as a result of the formation of a central cumulative jet.

The expansion of a toroidal gas bubble was studied in [1] for a compressible fluid. Results of this study make it possible to determine the maximum radius of the bubble, but in [1] it was assumed that the circular axis of the bubble is stationary. Furthermore, in a number of cases the expansion of the toroidal bubble is accompanied by its motion in the vertical and horizontal directions.

Consider the dynamics of a toroidal bubble, taking into account the displacement of its circular axis. Let the toroidal gas bubble, with radius of the circular axis a and radius of the cross section R_0, be initially in the horizontal plane in an unbounded incompressible fluid of density ρ. The fluid pressure inside the bubble is p_0, the fluid pressure at the depth of location of the bubble is p_∞, and the bubble has a vertical velocity V. The dynamics of the bubble includes two basic components: the pulsating motion near the circular axis due to the pressure difference $p_0 - p_\infty$ and the motion in the vertical direction due to the presence of vertical velocity and the buoyancy force.

In order to derive the equations of motion of the bubble we determine the velocity potential caused by the pulsating and buoyant torus, i.e., find the harmonic function outside the torus whose large and small radii are, respectively, a and R.

Starting from multipole series for the potential

$$\Phi = D_0\Phi_0 + \sum_{i=1}^{2} D_i \frac{\partial \Phi_0}{\partial z_i} + \frac{1}{2}\sum_{i,j=1}^{2} D_{ij}\frac{\partial^2\Phi_0}{\partial z_i \partial z_j} + \ldots, \tag{1}$$

where $\Phi_0 = \frac{2}{\pi}\frac{K(k)}{\sqrt{(z_1+a)^2+z_2^2}}$ is the potential of a single ring source; K is the complete elliptic integral of the first kind; $k = \sqrt{\frac{4az_1}{(z_1+a)^2+z_2^2}}$ is the modulus of the elliptic integral; z_1, z_2 are the cylindrical coordinates system whose origin is in the plane of the circular axis of the torus (z_1 is the radial coordinate and z_2 is the axial coordinate); D_0, D_i, and D_{ij} are the multipole moments.

Consider polar coordinates $z_1 = a + r\cos\theta$, $z_2 = r\sin\theta$. In order to compute the potential (1), the elliptic integrals are replaced by their asymptotes, which are valid for $r/a \ll 1$ and expressed through harmonic functions [2]. The potential is expressed in the form $\Phi = \Phi^{(+)} + \Phi^{(-)}$, where $\Phi^{(+)}$ is an even function in θ and represents the potential of the pulsating motion of the bubble; $\Phi^{(-)}$ is an odd function in θ and describes the potential of the forward motion of the torus. In order to determine the potential $\Phi^{(+)}$ let $D_i = R^2 d_i/a^2$, $D_{ij} = R^4 d_{ij}/a^4$; to compute $\Phi^{(-)}$ put $D_{ij} = R^2 d_{ij}/a^2$, $D_{ijk} = R^4 d_{ijk}/a^4$. The quantities D_0, d_i, d_{ij}, and d_{ijk} are found from kinematic conditions at the boundary of the torus:

$$\left.\frac{\partial\Phi^{(+)}}{\partial r}\right|_{r=R} = \dot{R},\quad \left.\frac{\partial\Phi^{(-)}}{\partial r}\right|_{r=R} = \dot{Z}\sin\theta,$$

where Z is the vertical distance of the bubble from its initial condition; here and in what follows, the dot over a symbol denotes differentiation in time.

Limiting ourselves to terms of the order R^2/a^2, we get

$$\Phi^{(+)} = -R\dot{R}\left\{\left[\Lambda + (2\Lambda - 3)\frac{r^2}{16a^2} + \frac{R^2}{8a^2}(\Lambda_R - 2)(2\Lambda - 1)\right]\right. \tag{2}$$

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$$-\left[(2\Lambda-1)\frac{r}{2a}+\frac{R^2}{2a^2}(\Lambda_R-2)\frac{a}{r}\right]\cos\theta+\left[(3\Lambda-4)\frac{r^2}{16a^2}+\frac{R^2}{8a^2}(\Lambda_R-2)+\frac{R^2}{32a^2}(6\Lambda_R-11)\frac{R^2}{r^2}\right]\cos 2\theta\Big\};$$

$$\Phi^{(-)}=-R\dot{Z}\left\{\left[\frac{R}{r}+\frac{R^2}{32a^2}(4\Lambda_R-7)\frac{R}{r}+\frac{R^2}{32a^2}(4\Lambda-3)\frac{r}{R}\right]\sin\theta-\frac{R}{4a}\sin 2\theta+\frac{R^2}{32a^2}\left(3\frac{r}{R}+\frac{R^3}{r^3}\right)\sin 2\theta\right\}, \quad (3)$$

where $\Lambda=\ln\frac{8a}{r}$; $\Lambda_R=\Lambda|_{r=R}=\ln\frac{8a}{R}$.

On the basis of Eqs. (2) and (3), the expression for the kinetic energy of the fluid may be expressed in the following manner:

$$T=-\pi aR\rho\int_0^{2\pi}\left(\Phi\frac{\partial\Phi}{\partial r}\right)\Big|_{r=R}d\theta$$

$$2\pi^2aR^2\dot{R}^2\rho\left[\Lambda_R+\frac{R^2}{16a^2}(4\Lambda_R^2-8\Lambda_R+1)\right]+\pi^2aR^2\dot{Z}^2\rho\left[1+\frac{R^2}{16a^2}(4\Lambda_R-5)\right]. \quad (4)$$

The expression for the potential energy of the system fluid-bubble has the form

$$\Pi=\frac{2\pi^2aR^2}{\varkappa-1}p_0\left(\frac{R_0}{R}\right)^{2\varkappa}+2\pi^2aR^2(p_\infty-\rho gZ),$$

where $\varkappa$ is the adiabatic index of the gas inside the bubble. Differentiating the energy balance equation T + Π = const with respect to time and adding the equation for the momentum in the vertical direction to the result [in computing the added mass of the fluid we take into account the second term in (4)], we obtain a system of equations describing the motion of the toroidal bubble:

$$2\dot{R}\left[\dot{R}^2\Lambda_R+\frac{1}{2}\dot{Z}^2+\frac{p_\infty}{\rho}-gZ-\frac{p_0}{\rho}\left(\frac{R_0}{R}\right)^{2\varkappa}\right]+R\Big(2\dot{R}\ddot{R}\Lambda_R+$$

$$+\dot{Z}\ddot{Z}-g\dot{Z}-\frac{\dot{R}^3}{R}\Big)+\frac{R^2}{16a^2}[4(2\Lambda_R-3)\dot{R}\dot{Z}^2+(4\Lambda_R-5)R\dot{Z}\ddot{Z}+2(4\Lambda_R^2-8\Lambda_R+$$

$$+1)R\dot{R}\ddot{R}+4(4\Lambda_R^2-10\Lambda_R+3)\dot{R}^3]=0; \quad (5)$$

$$(2\dot{R}\dot{Z}+R\ddot{Z})\left[1+\frac{R^2}{16a^2}(4\Lambda_R-5)\right]+\frac{R^2}{8a^2}(4\Lambda_R-7)\dot{R}\dot{Z}=gR. \quad (6)$$

Introducing nondimensional variables $R'=R/R_0, Z'=Z/R_0, t'=\frac{t}{R_0}\sqrt{\frac{p_\infty}{\rho}}$, and after the transformations, Eqs. (5) and (6) can be written in a form that is convenient for numerical integration:

$$\dot{u}=\frac{1}{R\Lambda_R}\left\{\left(\delta R^{-2\varkappa}+\frac{u^2+v^2}{2}+\beta Z-u^2\Lambda_R-1\right)\left[1-\right.\right.$$

$$\left.-\frac{R^2}{16a^2}\frac{(4\Lambda_R^2-8\Lambda_R+1)}{\Lambda_R}\right]+\frac{R^2}{8a^2}\left[(2\Lambda_R-3)v^2-(4\Lambda_R^2-10\Lambda_R+3)u^2\right]\Big\},$$

$$\dot{v}=\beta-\frac{2uv}{R}-\frac{R}{16a^2}[(4\Lambda_R-5)\beta R+2(4\Lambda_R-7)uv],\ \dot{R}=u,\ \dot{Z}=v; \quad (7)$$

$$R=1,\ Z=0,\ u=\varepsilon,\ v=\alpha \text{ for } t=0.$$

In Eq. (7): $\beta=\frac{\rho gR_0}{p_\infty}$, $\delta=\frac{p_0}{p_\infty}$, $\alpha=\frac{V}{\sqrt{p_\infty/\rho}}$, $\varepsilon=\frac{U}{\sqrt{p_\infty/\rho}}$, where U is the initial value of $\dot{R}$. Here and in what follows primes with nondimensional terms are omitted. Results for the expansion of a toroidal bubble with a_0 = 500, 5000, and 50,000 (lines 1-3) are shown in Fig. 1, where $\alpha=\mu=0$, $\beta=10^{-3}$, $\delta=10^4$, $\varepsilon=1$, $\varkappa=4/3$.

Consider the collapse of the empty bubble $\delta=0$ when $\beta\ll 1$ in the approximation of a slender torus: In Eqs. (5) and (6) terms of the order R^2/a^2 are neglected. Then the basic equations could be reduced to energy and momentum equations

$$R^2\left(\dot{R}^2\Lambda_R+\frac{1}{2}\dot{Z}^2+1\right)=\frac{1}{2}\alpha^2+1,\ R^2\dot{Z}=\alpha. \quad (8)$$

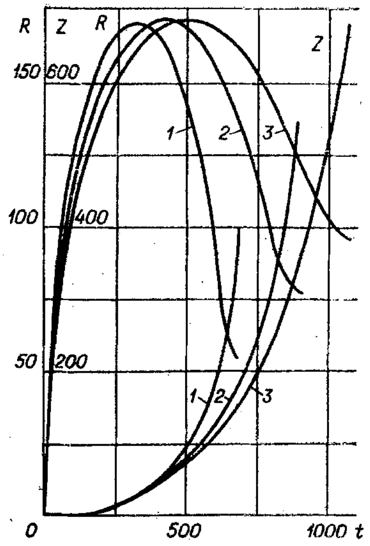

Fig. 1

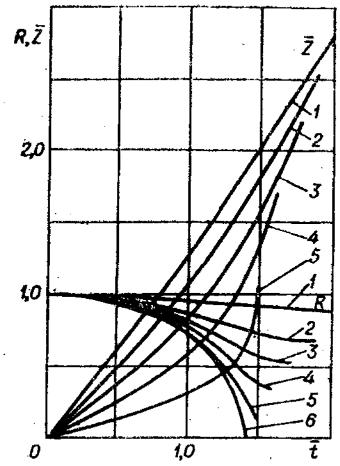

Fig. 2

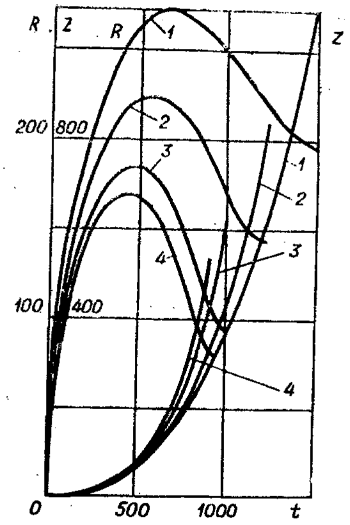

Fig. 3

It follows from (8) that

$$\dot{R} = -\frac{1}{\sqrt{\Lambda_R}} \frac{\sqrt{(1-R^2)(R^2-R_*^2)}}{R^2}, \tag{9}$$

where $R_* = \alpha/\sqrt{2}$ is the minimum radius of the bubble corresponding to the condition $\dot{R} = 0$. The minus sign for the square root is chosen from the condition for the consideration of the stage of collapse. Note that when $R = R_*$ the maximum vertical speed of the bubble is attained, which is determined from the equation $\max|\dot{Z}| = \sqrt{2}/R_*$. Equation (9) leads to the quadrature

$$t = \int_R^1 \sqrt{\Lambda_R} \frac{R^2 dR}{\sqrt{(1-R^2)(R^2-R_*^2)}}.$$

For an approximate computation of the integral we replace the variable of integration within the logarithmic sign by unity. Then we get the relation

$$t = \sqrt{\ln 8a}\,E(\varphi, k_+),\ Z = \alpha\sqrt{\ln 8a}\,F(\varphi, k_+), \tag{10}$$

where F and E are incomplete elliptic integrals of the first and second kind, respectively;

$$k_+ = \sqrt{1-R_*^2};\ \varphi = \arcsin\sqrt{\frac{1-R^2}{1-R_*^2}}.$$

Results of the calculations according to formula (10) are shown in Fig. 2, where dependence R is given and the introduced vertical displacement of the bubble $\bar{Z} = Z\sqrt{2/\ln 8a}$ derived from introduced time $\bar{t} = t\sqrt{2/\ln 8a}$ when $\alpha = 1.25$; 1.0; 0.75; 0.5; 0.25; in the case of 0 (lines 1-6), $\beta = \delta = \varepsilon = \mu = 0$.

Equations for the collapse time and for the corresponding vertical advance follow from Eq. (10):

$$t_* = \sqrt{\ln 8a}\,E(k_+),\ Z_* = \alpha\sqrt{\ln 8a}\,K(k_+), \tag{11}$$

where E is the complete elliptic integral of the second kind.

Results obtained from Eqs. (10) and (11) were compared with computed results from the solution to Eq. (7) assuming a slender torus; as a result it is established that the error in approximate analytical expressions does not exceed 5%.

When $\alpha = 0$, the equation for the collapse of an empty, stationary bubble, assuming a slender torus, agrees with the corresponding equation from [1]:

$$R\ddot{R}\Lambda_R + (\Lambda_R - 1/2)\dot{R}^2 + 1 = 0.$$

An approximate solution to this equation ($\Lambda_R \to \sqrt{\ln 8a}$) has the form

$$R = \sqrt{1 - t^2/\ln 8a}.$$

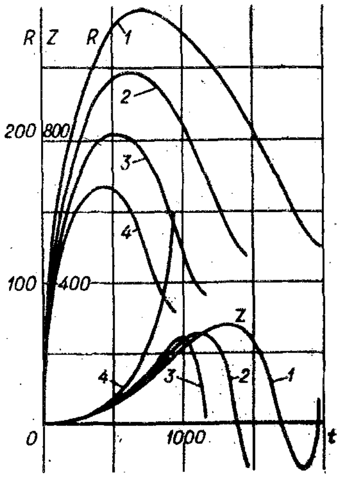

Fig. 4

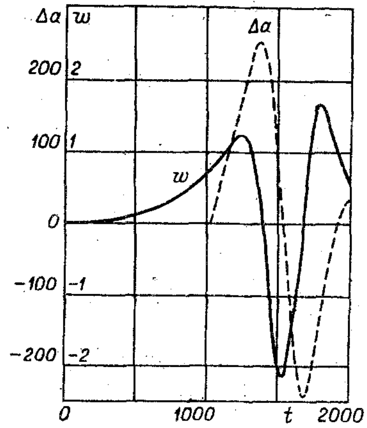

Fig. 5

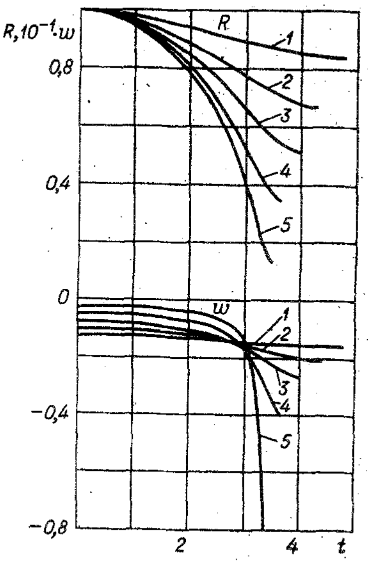

Fig. 6

The formation of a cumulative jet can lead to the formation of circulatory motion around the circular axis of the torus and to the appearance of the horizontal velocity $\dot{a} = w$.

The presence of circulation Γ leads to the appearance of Zhukovskii lift force on the bubble with vertical and horizontal components. In this case, in addition to the pulsatory and vertical motions of the bubble there could be a displacement of its circular axis in the horizontal plane. The problem of the motion of the toroidal bubble in the presence of circulation approaches the problem of the vortex ring lift considered in [3] without pulsatory motion.

Consider the equations for kinetic energy, momentum, and velocity of the vortex ring given in [4] assuming a slender torus. The justification for using these relations with these assumptions is given in [5]. Then the energy equation may be written in the following manner:

$$aR^2\left(\dot{R}^2\Lambda_R + \frac{v^2+w^2}{2}\right) + \frac{1}{4}\gamma^2 a(\Lambda_R - 7/4) + \frac{\delta}{\varkappa - 1} aR^{2(1-\varkappa)}\left(\frac{a_0}{a}\right)^{\varkappa} + aR^2(1-\beta Z) = \text{const}, \tag{12}$$

where $\gamma = \Gamma/(\pi R_0\sqrt{p_\infty/\rho})$.

Equations for the vertical and horizontal momentum of the bubble have the form

$$(aR^2v)^{\cdot} = \beta aR^2 - \gamma aw,\quad (aR^2w)^{\cdot} = \gamma av.$$

The right-hand side of these equations represent buoyancy and Zhukovskii lift forces acting on the bubble. The first equation for $\beta = 0$ has an integral $aR^2v = (1/2)\gamma(a_0^2 - a^2) + \alpha a_0$.

After differentiating of Eq. (12) with respect to time and performing some transformations, the basic system of equations describing the dynamics of the toroidal bubble in the presence of circulation is reduced to the form

$$\dot{u} = \frac{1}{2u\Lambda_R}\left\{\left(\frac{2u}{R} + \frac{w}{a}\right)\left[\delta\left(\frac{a_0}{a}\right)^{\varkappa} R^{-2\varkappa} + \frac{v^2+w^2}{2} + \beta Z - u^2\Lambda_R - 1\right]\right.$$
$$\left. + \beta\gamma\frac{1}{4\pi a}(\Lambda_R - 1/4) + \frac{\gamma^2}{4aR^2}\left[\frac{au}{R} - w(\Lambda_R - 3/4)\right] + u^2\left(\frac{u}{R} - \frac{w}{a}\right)\right\},$$
$$\dot{v} = \beta - \frac{2uv}{R} - \frac{vw}{a} - \frac{\gamma w}{R^2},\quad \dot{w} = \frac{\gamma v}{R^2} - \frac{2uv}{R} - \frac{w^2}{a},\quad \dot{R} = u, \tag{13}$$
$$\dot{Z} = v + \frac{\gamma}{4\pi a}(\Lambda_R - 1/4),\quad \dot{a} = w.$$

When $t = 0$ $R = 1$, $a = a_0$, $Z = 0$, $\dot{R} = \varepsilon$, $\dot{Z} = \alpha$, $\dot{a} = \mu$, where $\mu = W/\sqrt{p_\infty/\rho}$, W is the initial value of $\dot{a}$.

Results of numerical solution to the problem of the motion of the bubble under the action of gaseous products contained in it are given in Figs. 3-5 for different values of μ and γ ($a_0 = 10^4$, $\beta = 10^{-3}$, $\delta = 10^4$, $\varepsilon = 1$, $\varkappa = 4/3$): in Fig. 3 ($\alpha = \gamma = 0$) lines 1-4 denote μ = 300; 200; 100; and 0; in Fig. 4 ($\alpha = \mu = 0$) lines 1-4 denote γ = 200; 150; 100; and 0; in

Fig. 5 $\Delta a = a - a_0$, $\alpha = \mu = 0$, $\gamma = 150$. Computations show that when $\gamma \neq 0$, the displacement of the circular axis may have an oscillatory character.

When $\beta = \gamma = 0$, Eq. (13) may be written as a law of conservation of energy and momentum:

$$\sqrt{\Lambda_R}\, aR^2\dot{R} = \pm\sqrt{(a_0 - aR^2)(aR^2 - a_0R_\Delta^2) + \frac{a_0\delta}{\varkappa - 1}\left[aR^2 - a_0^{\varkappa-1}(aR^2)^{2-\varkappa}\right]},$$
$$aR^2\dot{a} = \mu a_0,\ aR^2\dot{Z} = \alpha a_0, \qquad (14)$$

where $R_\Delta = \sqrt{(\alpha^2 + \mu^2)/2}$. The signs + and – in front of the radical correspond to expansion and collapse of the bubble. The last two equations (14) lead to an explicit relation $Z = \alpha(a - a_0)/\mu$ which makes it possible to eliminate the equation for Z from further analysis. Equating the expression within the radical to zero, we obtain a relation between the parameters of the problem when the bubble radius has extremal values.

In the case of the collapse of the bubble ($\delta = 0$, $R < 1$) the first two equations of the system (14) describe the motion of the bubble between the states a_0 and $a_0R_\Delta^2$. If $\mu > 0$, $R_\Delta > 1$ then $a_0 \leqslant aR^2\ \ a_0R_\Delta^2$; if $\mu < 0$, $R_\Delta < 1$ then $a_0R_\Delta^2 \leqslant aR^2 \leqslant a_0$. When minimum radius R_Δ is attained, the bubble size and speed are determined from

$$\dot{a}\Big|_{R=R_\Delta} = \frac{\mu}{R_\Delta^2},\ \dot{Z}\Big|_{R=R_\Delta} = \frac{\alpha}{R_\Delta^2}.$$

Computed results for the system (13) in the case of the collapse of the empty bubble are given in Fig. 6 where $\alpha = \beta = \gamma = \delta = \varepsilon = 0$, $a_0 = 10^4$ and lines 1-5 correspond to $\mu = 1.25, -1, -0.75, -0.5$, and -0.25.

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INVESTIGATION OF HEAT TRANSFER IN SEPARATED REGIONS IN A SUPERSONIC LAVAL NOZZLE

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1. This paper describes experiments and an approximate method to compute heat transfer at the wall in supersonic flow with separation in the presence of a step-cavity profile of the divergent part of the Laval nozzle. Tests were conducted on a horizontal jet facility in which a plane nozzle with cavities (Fig. 1a) was mounted. Local heat transfer and pressure coefficients at the nozzle walls, including the separate region, were measured. Special thin film gauges [1] were used to measure heat transfer coefficients under complex flow conditions. The flow parameters were as follows: stagnation temperature T_0 = 250-270°K, total pressure $p_0 = (9.0\text{-}12.5)\cdot 10^5$ Pa, and Reynolds number based on the throat section dimension, $Re = 6\cdot 10^6$. Measurements were made on three models of the nozzle with Mach numbers at the separation point at the edge of the cavity: 1.90, 2.28, and 2.61, respectively. A schematic diagram of the cavity is shown in Fig. 1a. The length of the horizontal wall of the cavity had the following values: L = 0, 14, 27, 47, and 60 mm. The height of the wall h = 12 mm was kept constant in all the tests and was appreciably more than the boundary layer thickness ahead of separation (δ_1/h = 0.13-0.17).

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